

Assignment 4.

Polynomials, exponential, related functions.

This assignment is due Wednesday, Feb 18. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

- (1) (a) Prove that $z = z_0$ is a zero of multiplicity k of the polynomial $f(z)$ if and only if

$$f(z_0) = f'(z_0) = \dots = f^{(k-1)}(z_0) = 0, \quad f^{(k)} \neq 0.$$

(*Hint:* It was proved in class that a zero $z = z_0$ of $f(z)$ has multiplicity k iff it is of multiplicity $k - 1$ as a zero of $f'(z)$. Use this.)

- (b) Prove that if $z_0 = a + bi$ is a zero of multiplicity k of the polynomial $f(z)$ with real coefficients, then $\bar{z}_0 = a - bi$ is also a zero of multiplicity k of $f(z)$. (*Hint:* Conjugate the formula in the item 1a above.)
- (c) Prove that every polynomial with real coefficients decomposes as a product of linear and quadratic polynomials with real coefficients. (*Hint:* Look at $(z - z_0)(z - \bar{z}_0)$.)
- COMMENT. This is one of many examples where the fact purely about real numbers is proved using complex numbers.
- (d) Decompose $x^4 + 1$ in a product of two polynomials with real coefficients.

- (2) Give another proof of existence and uniqueness of the exponential function (Theorem 10 in Sec. 4.2 of lecture notes), following the steps below:

- (a) Prove that $f'(0) = 1$. (*Hint:* Argue why the limit $\lim_{\Delta z \rightarrow 0} \frac{f(\Delta z) - f(0)}{\Delta z}$

must be the same as $\lim_{\Delta x \rightarrow 0} \frac{f(\Delta x) - f(0)}{\Delta x}$, $x \in \mathbb{R}$.)

- (b) Prove that $f'(z) = f(z)$ for all $z \in \mathbb{C}$. (*Hint:* Use the limit of ratios definition of derivative and the result above.)
- (c) Treating the resulting differential equation $f' = f$, $f(0) = f'(0) = 1$ as a system of real-number differential equations, prove that the only complex differentiable function f that satisfies it is $f(z) = e^x(\cos y + i \sin y)$. (*Hint:* For example, if $f = u + iv$, you get $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$, so $u = \frac{\partial u}{\partial x}$ and $v = \frac{\partial v}{\partial x}$. Don't forget Cauchy–Riemann equations.)

- (3) It is known from calculus that $\lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n = e^x$ ($x \in \mathbb{R}$). Follow the steps below to prove that $\lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n = e^z$ for any $z \in \mathbb{C}$.

- (a) Fix a complex number z . Denote $z_n = (1 + \frac{z}{n})^n$. Find $\lim_{n \rightarrow \infty} |z_n|$.

Hint: Note that $|z_n| = |1 + \frac{x}{n} + i \frac{y}{n}|^n = \left((1 + \frac{x}{n})^2 + \frac{y^2}{n^2} \right)^{n/2}$. Argue that you can discard terms with $1/n^2$; for example, by showing that

the limit $\lim_{n \rightarrow \infty} \frac{\left((1 + \frac{x}{n})^2 + \frac{y^2}{n^2} \right)^{n/2}}{\left(1 + \frac{2x}{n} \right)^{n/2}}$ is equal to 1.

- (b) Find $\lim_{n \rightarrow \infty} \text{Arg } z_n$. (In particular, pick appropriate values of Arg for each n so that the limit exists.)

Hint: Show that $n \arctan \frac{y/n}{1+(x/n)} \in \text{Arg } z_n$. Argue that you can replace \arctan with its argument as $n \rightarrow \infty$; for example, by showing that

$$\lim_{n \rightarrow \infty} \frac{\arctan \frac{y/n}{1+(x/n)}}{\frac{y/n}{1+(x/n)}} = 1.$$

- (c) Combine results of (a) and (b) to get $\lim_{n \rightarrow \infty} z_n$.

— see next page —

- (4) (a) (*Logarithmic spiral.*) Find and sketch the image of a straight line

$$z = (1 + i\alpha)t + ib, \quad -\infty < t < +\infty,$$

$\alpha, b \in \mathbb{R}$, $\alpha \neq 0$, under the map $w = e^z$. Eliminate t in the answer to get an equation in polar coordinates. (This straight line has slope α (meaning it intersects x -axis at the angle $\arctan \alpha$) and intersects y -axis at the point ib .)

(*Hint:* After eliminating t , the answer should be $r = ce^{\varphi/\alpha}$, where $c = e^{-b/\alpha}$. This curve is called a *logarithmic spiral*.)

- (b) Use conformity of $w = e^z$ to find the angle at which logarithmic spiral intersects rays emanating from the origin.
- (c) Given $\alpha \in \mathbb{R}$, find the condition on real numbers $c, c' > 0$ for the spirals $r = ce^{\varphi/\alpha}$ and $r = c'e^{\varphi/\alpha}$ to be the same (as sets on a plane).
COMMENT. In other words, logarithmic spiral is *self-similar*.
- (d) (*Optional, i.e. not included in denominator.*) Find and sketch the image under $w = e^z$ of the slanted strip S with sides

$$y = k(x - a_1), \quad y = k(x - a_2) \quad (k, a_1, a_2 \in \mathbb{R}; k \neq 0).$$

(*Hint:* Convert equations of straight lines to the form used in item 4a.)

- (e) (*Optional, i.e. not included in denominator.*) Find the condition on k, a_1, a_2 for the map $\exp : S \rightarrow \mathbb{C}$ to be injective, (i) using item 4c above, (ii) using periodicity of \exp .

- (5) (a) Find $(\cos z)'$, $(\sin z)'$, $(\sinh z)'$, $(\cosh z)'$. (Probably, the quickest way is to express these functions through \exp .)
- (b) Find all complex solutions of equations $\cos z = 0$, $\sin z = 0$, $\cosh z = 0$, $\sinh z = 0$.
- (c) Using formulas for $\cos(z_1 + z_2)$ and $\sin(z_1 + z_2)$, express $|\cos z| = |\cos(x + iy)|$ and $|\sin z| = |\sin(x + iy)|$ through \sin, \cos, \sinh, \cosh of x and y .